Decoherence in chaotic and integrable systems: A random matrix approach

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Abstract. – In order to analyze the effect of chaos or order on the rate of decoherence in a subsystem we aim to distinguish effects of the two types of dynamics from those depending on the choice of the wave packet. To isolate the former we introduce a random matrix model that permits to vary the coupling strength between the subsystems. The case of strong coupling is analyzed in detail, and we find at intermediate times a weak effect of spectral correlations that is reminiscent of the correlation hole.

New experimental techniques in atomic and quantum optics and more recently in solid state physics have made measurements of decoherence of entangled states possible. The perspective of quantum computing makes this subject also relevant to applications. In this context the question arises how the integrability or chaoticity of the corresponding classical systems *i.e.* "quantum chaos" affects the process of decoherence [1–3]. Such properties manifest themselves both in the spectrum and the wave functions. While the former is invariant the latter are basis dependent. Yet that does not mean that the latter are irrelevant in a semi-classical context; indeed any wave packet localized in phase space will feel special features of the dynamics such as KAM tori or short periodic orbits much more strongly, than their influence on the spectrum. Such effects will be more pronounced in integrable or near integrable systems, than in chaotic ones, because KAM tori are felt everywhere. If we think of the possible configuration of systems this can refer both to the initial wave packet and to the Hamiltonian. For the former we can *e.g.* think of successive laser excitations in Rydberg systems, and for the latter of appropriate external fields in atoms or of designed mesoscopic systems. It is therefore relevant to ask which properties are due to the nature of the system, and which are due to the preparation of the packet. Studying decoherence we can never do entirely without a packet, but to reduce the influence of preparation to a minimum, random matrix theory (RMT) is ideally suited, and we will develop such models for a wide range of situations.

To construct our RMT model we start from the standard assumption, that the classical ensembles [4] (such as the Gaussian orthogonal ensemble (GOE) for time reversal invariant systems) describe the universal features induced by classical chaos in quantum systems [5]. For the classically integrable situation we expect a random spectrum if we exclude harmonic oscillators [6]. For the description of a random wave packet the orthogonal invariance of

the GOE is the ideal tool. This concept has been extended to the integrable case by the introduction of the Poisson orthogonal ensemble (POE) which combines a random spectrum with orthogonal invariance [7]. These models serve well to distinguish effects of the invariant properties embodied in spectral fluctuations, from others depending on the choice of a specific basis or wave packet as well as on the level density.

Decoherence will be treated in the framework of unitary time evolution and partial traces over subsystems outlined in [3]. The model we develop allows to analyze various relevant situations, but we limit explicit analytic and numeric calculations in this letter to a strong coupling limit in which the preparation of the wave packet and the separation into subsystems are completely unrelated to the Hamiltonian. This limit covers some physical situations [8], though not the case originally discussed by Zurek [1].

To visualize our problem consider a Hamiltonian consisting of three terms $H = H^{(0)}+V^{(1,2)}$ with $H^{(0)} = h^{(1)} + h^{(2)}$ where the two terms of $H^{(0)}$ act on different degrees of freedom of the system, while $V^{(1,2)}$ is an interaction. Note that the $h^{(i)}$ may act on one or several degrees of freedom each, and $V^{(1,2)}$ may or may not induce chaos. Indeed the total system may be integrable and separable in a different set of coordinates. Consider *e.g.* atoms or molecules coupled to the radiative field, spin degrees of freedom coupled to orbital ones or coupled cavities with fields.

If we consider this Hamiltonian as a quantum operator we shall denote by $\mathcal{H}_1$ and $\mathcal{H}_2$ the Hilbert spaces on which $h^{(1)}$ and $h^{(2)}$ respectively act. The total Hamiltonian H acts on the product space $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. We write the basis states of $\mathcal{H}_1$ as kets with Latin letters such as $|i\rangle$ and those of $\mathcal{H}_2$ as kets with Greek letters such as $|\mu\rangle$. The states $|i, \mu\rangle = |i\rangle|\mu\rangle$ with indices conveniently written as pairs, form an eigenbasis of $H^{(0)}$. H is diagonal in a different basis, which we characterize by a single index α, $\alpha = 1 \ldots \bar{N}$. Thus $H_{i\mu,i'\mu'} = \sum_\alpha O_{i\mu,\alpha}\, E_\alpha\, O^\tau_{\alpha,i'\mu'}$ where E_α denotes elements of the diagonal energy matrix E and O the orthogonal transformation between the two bases.

The Hamiltonians $h^{(1)}$ and $h^{(2)}$ and the interaction term may be taken from ensembles or fixed according to the situation we wish to analyze. We distinguish between strong and weak interaction. The interaction strength is usually discussed in terms of the spreading width $\Gamma^{(0)}$, which indicates the width of the distribution of the expansion coefficients of the eigenstates of H in terms of that of $H^{(0)}$. To study the behaviour of an initial state, that is a product of a state in $\mathcal{H}_1$ with another in $\mathcal{H}_2$, we have to compare the width Γ of this initial state in the eigenbasis with $\Gamma^{(0)}$. Typically we speak of strong coupling if many states are mixed with large amplitudes, and of weak coupling if the states essentially retain their identity with small admixtures of other states which permits a perturbative treatment. In the strong coupling case we can always choose a wave packet such that $\Gamma \ll \Gamma^{(0)}$ and we shall include this assumption in the term "strong coupling" throughout this paper. In the weak coupling case we always have $\Gamma \gg \Gamma^{(0)}$ because Γ must contain at least four states for a non-trivial decoherence effect. Note that the last condition is not restricted to weak interactions, and intermediate situations can occur.

We now construct appropriate matrix ensembles for the three terms in the Hamiltonian. The case of weak coupling will not be discussed in detail but we may mention, that $h^{(2)}$ could describe the central system and $h^{(1)}$ the environment, *e.g.* the heat bath, or a finite quality resonator. In any case both $h^{(1)}$ and $h^{(2)}$ could pertain to either of the above ensembles, and $V^{(1,2)}$ would typically be symmetric with independent Gaussian distributed matrix elements. The case studied by Zurek could be simulated using a GOE with high level density for $h^{(1)}$ and a GOE or a POE respectively for $h^{(2)}$. A detailed perturbation study of weak coupling cases will be published elsewhere.

In the case of strong coupling $h^{(1)}$ and $h^{(2)}$ determine the factor spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

Their spectral properties are irrelevant except for their relative spectral density in the energy region where the wave packet lives. The interaction $V^{(1,2)}$ will be given by the GOE for chaotic systems and by the Poisson orthogonal ensemble (POE) [7] for integrable ones. Both ensembles are given by matrices of the form OEO^τ, where E is a diagonal energy matrix, and O is a orthogonal matrix distributed according to the Haar measure of the orthogonal group. For the GOE the distribution of the energies has complicated correlations and a semi circle density, while they are independently Gaussian distributed for the POE. The case of strong coupling both for integrable and chaotic systems was modeled in [8] with two-dimensional anharmonic oscillators, and the systems considered in [3] might be close to this domain.

We expect the spectral correlations of the ensembles to represent those of the corresponding Hamiltonian systems well. The same is not true for the level densities, which are determined by the phase space volume. Yet we are mainly interested in a small energy region in which the wave packet is concentrated, and therefore we shall assume a constant level density. The effect of rapidly changing level density is within the scope of this model. In the above representations of our ensembles we therefore replace the energies by unfolded energies with the same spectral fluctuations, so that the local average spacing becomes constant. We shall also use equidistant "picket fence" spectra to complete the range of possible spectral correlations. The latter is important, because both oscillators in many dimensions and low dimensional systems have spectra that are much stiffer [9] than what we expect from the universal random matrix properties of integrable systems [7].

Wave packets, while fluctuating in energy space, usually have a smooth envelope. We replace it by a sharp cutoff as follows: The packet has n components in $\mathcal{H}_1$ and m in $\mathcal{H}_2$, where n/m is determined by the relative spectral density of the two subsystems in the energy region we consider, and $n \times m = N$ is determined by the width $\Gamma = Nd$, where d is the local mean level spacing. We consider the Hilbert space made up of the tensor product of the spaces spanned by these vectors leading to a $N < \bar{N}$ dimensional total space. This approximation is drastic, but it is important to note that we only restrict the shape of the packet in energy space and *not* the fluctuations. The way we truncate the space is consistent with the original structure, in that the truncated packet is again a product of functions in the two subspaces, and applying the same notation we used for the infinite dimensional subspaces to the finite dimensional ones, we still have $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$.

Consider $\rho_{i\mu,i'\mu'}$ to be a density matrix constructed from a "product state" *i.e.* is pure with respect to both pairs of indices. With other words we have $\mathrm{Tr}_1[\mathrm{Tr}_2(\rho)]^2 = \mathrm{Tr}_2[\mathrm{Tr}_1(\rho)]^2 = 1$. Here Tr_1 indicates the trace with respect to the first (Latin) index and Tr_2 the one over the second (Greek) index. We shall choose a density matrix $\rho = \rho(0)$ fulfilling this condition as initial state, watch its time evolution and finally obtain ensemble averaged properties of partial traces. The simplest quantity to analyze is the purity defined as

$$I(t) = \mathrm{Tr}_1[\mathrm{Tr}_2(\rho(t))]^2 \;=\; \mathrm{Tr}_2[\mathrm{Tr}_1(\rho(t))]^2. \tag{1}$$

We are free to interchange summations as all sums are finite. The definition of the purity I is related to the idempotency defect or linear entropy, defined as $1 - I$ [3].

Denoting by Δ the diagonal matrix with entries $\Delta_\alpha = \exp[it\,E_\alpha]$ we find in the basis of double indices $\rho(t) = O\Delta O^\tau \rho(0) O\Delta^* O^\tau$ and by consequence

$$I(t) = \mathrm{Tr}_1[\mathrm{Tr}_2(O\Delta O^\tau \rho(0) O\Delta^* O^\tau)\mathrm{Tr}_2(O\Delta O^\tau \rho(0) O\Delta^* O^\tau)] \tag{2}$$

To form the ensemble average of this quantity we use that the measure factorizes. We take the averages involving energies and the averages involving states separately for different terms of the sum. The only object that will not be averaged over is the original pure density matrix

$\rho(0)$. In principle we can perform the averages and then use the idempotency condition, but as we average over all orthogonal transformations we can reinterpret the states $|i\rangle|\mu\rangle$ as pertaining to a basis where $\rho(0)$ is diagonal with $\rho(0)_{11,11} = 1$ and all other matrix elements zero; it is easy to verify that this is possible within the product basis. Using this form of $\rho(0)$ we obtain for the two averages:

$$AE_{\alpha,\beta;\gamma,\delta} = \langle \Delta_\alpha \Delta_\gamma \Delta^*_\beta \Delta^*_\delta \rangle = \langle \exp[\mathrm{i}t(E_\alpha + E_\gamma - E_\beta - E_{\delta)} \rangle \tag{3}$$

$$AO_{\alpha,\beta;\gamma,\delta} = \sum_{\mu,\nu,i,j} \langle O_{i\mu,\alpha} O_{11,\alpha} O_{11,\beta} O_{j\mu,\beta} O_{j\nu,\gamma} O_{11,\gamma} O_{11,\delta} O_{i\nu,\delta} \rangle \tag{4}$$

The averages are connected only because one may force indices to be equal and thus reduce the other to a special case; as we shall see below five different terms exist.

We will not need all averages over monomials of eight matrix elements of the orthogonal group, as we concentrate on the relevant time scales. The ones we will need can be evaluated with Ullah's two-vector formula, if we correct an error therein [10]. We therefore consider the possible time scales in our problem, namely the inverse of the width Γ of the packet in energy space and the Heisenberg time $1/d$ where d is the mean level spacing. We have to inspect the effect of these quantities when folded on a circle by the exponential $\exp(\mathrm{i}t\, E_\alpha)$. We obtain four time scales for the evolution of the wave packet:

1) Short times where $t \ll 2\pi/\Gamma$, and thus perturbation theory applies. We will find the expected t^2 dependence with a factor given to leading order by the spreading width.

2) First filling of the unit circle, where $t = 2\pi/\Gamma$, and we will find a quadratic minimum for the purity with value $(2\pi/\Gamma) = 1/n + 1/m + 0(1/N)$.

3) Long times, where the spectrum winds many times around the unit circle as $1/d \gg t \gg 2\pi/\Gamma$, the winding acts as a random number generator and eliminates correlations to yield a result similar to the one for 2) though sub-leading terms may be different.

4) Poincaré recurrence time $t = 2\pi/d$, where a picket fence spectrum will cause exact recurrence, while even for GOE type correlations the recurrence is essentially wiped out. Yet for low-dimensional systems with their long-range stiffness [9] and for models involving harmonic oscillators this part may well be important. The same holds if the wave packet is extremely narrow.

We now proceed to derive and illustrate the results for the above. We evaluate the averages AE and AO starting with the former. There we deal with four indices and the result does not depend on the values of the indices. It depends only on whether certain indices are equal or not. If two indices of the energies coincide, we get either 0 if they have opposite signs or twice the energy of their signs are equal. It may readily be seen that five terms are possible:

$$\begin{aligned}
S_1(t) &= \langle \exp[-\mathrm{i}t\ (E_1 - E_2 + E_3 - E_4)] \rangle &&= f^4(t) \\
S_2(t) &= \langle \exp[-\mathrm{i}t\ (E_1 - E_2)] \rangle &&= f^2(t) \\
S_3(t) &= \langle \exp[-\mathrm{i}t\ (2E_1 - E_2 - E_3)] \rangle &&= f(2t) f^2(t) \\
S_4(t) &= \langle \exp[-\mathrm{i}t\ 0] \rangle &&= 1 \\
S_5(t) &= \langle \exp[-2\mathrm{i}t\ (E_1 - E_2)] \rangle &&= f^2(2t)
\end{aligned} \tag{5}$$

The result for each term corresponds to uncorrelated energies (POE). Here $f(t) = \sin(\Gamma t)/(\Gamma t)$ is the Fourier transform of the level density, which we assumed to be constant. For GOE

spectra the evaluation is more difficult, but some general considerations hold for any kind of spectrum. For long times all terms except S_4 go to zero. For short times, on the other hand, S_1 dominates because it has the largest weight. We now consider the four time regimes:

In the short time limit we expand the exponential. Due to the symmetry of the energy distribution linear terms in t vanish and the quadratic ones survive. These are of two types. Each exponential associated with a given index has a quadratic term, and indices in the linear terms of two exponentials may coincide. This implies that we only need the well-known averages over monomials of fourth order in the group elements to obtain

$$I(t) = 1 - 2\langle E_\alpha^2\rangle\, t^2[1 - (n+m+1)/(N+2)]\ . \tag{6}$$

In the last factor we seem to have a $1/N$ correction. Yet if n and m grow as $\sqrt{N}$ the correction is of order $1/\sqrt{N}$. If one of the two dimensions is kept constant, the other becomes proportional to N, and the second term even more important. Terms resulting from correlations of the energies are truly of order $1/N$ and were omitted.

The next time scale is that of the first filling of the unit circle, for which the minimum of the function $f^2(t)$ is reached. We have a complicated interplay of different terms and it seems that we need AO completely. To avoid this we can use a trick to obtain the answer at time $2\pi/\Gamma$. For uniform density of the spectrum the energy eigenvalues multiplied by $2\pi/\Gamma$ are essentially the eigenphases of a circular ensemble. For the case of GOE fluctuations the corresponding ensemble is known as the circular orthogonal ensemble (COE) [4], which is the ensemble of unitary symmetric $N \times N$ matrices S. This ensemble has a unique invariant measure. The only approximation we make is that we miss the correlations among the first and last levels that exist in a circular ensemble. In terms of S we obtain

$$I(2\,\pi/\Gamma) = \langle Tr_1[Tr_2(S\,\rho(0)\,S^*)Tr_2(S\,\rho(0)\,S^*)\rangle\ . \tag{7}$$

where $S_{i\mu,j\nu} = \sum_\alpha O_{i\mu,\alpha}\exp[\mathrm{i}E_\alpha(2\pi/\Gamma)]O_{j\nu,\alpha}$. The ensemble average is thus given in terms of averages over four symmetric unitary COE matrices, two of which are complex conjugate. Such averages are calculated in ref. [11] and we obtain

$$I(2\pi/\Gamma) = \frac{(n+m)N^2 + [3(n+m)+2]\,N - 2(n+m-1)}{N(N+1)(N+3)}\ . \tag{8}$$

As we shall see below this is slightly lower than the long time limit, while for the integrable (POE) case the long time limit and the value at $t = (2\pi/\Gamma)$ coincide.

In the long time limit $\Gamma t \gg 2\pi$, the process of stretching and taking modulo 2π is a reasonably efficient randomizer for a fluctuating set of numbers with correlations such as a GOE spectrum. Therefore the eigen-phases on this time scale are random both for the GOE and the POE. Thus only the term S_4 survives, where the indices of energies in conjugate terms coincide. The energy dependence, and therefore the time dependence, drops out and we are left with averages over the orthogonal group, AO. Only two-vector terms *i.e.* averages over elements from two rows of the matrix occur and we find

$$I_\infty = \frac{(n+m)N^3 + 3[4(n+m)+3]N^2 + [35(n+m)+57]N + 48}{(N+1)(N+2)(N+4)(N+6)}\ . \tag{9}$$

For the POE this result holds equally at time $t = 2\pi/\Gamma$, which we have discussed above, though it will differ for times near this one.

While the short-time behaviour is independent of the spectral statistics, at the first minimum we find a difference between GOE and POE. We may ask, what happens for very stiff

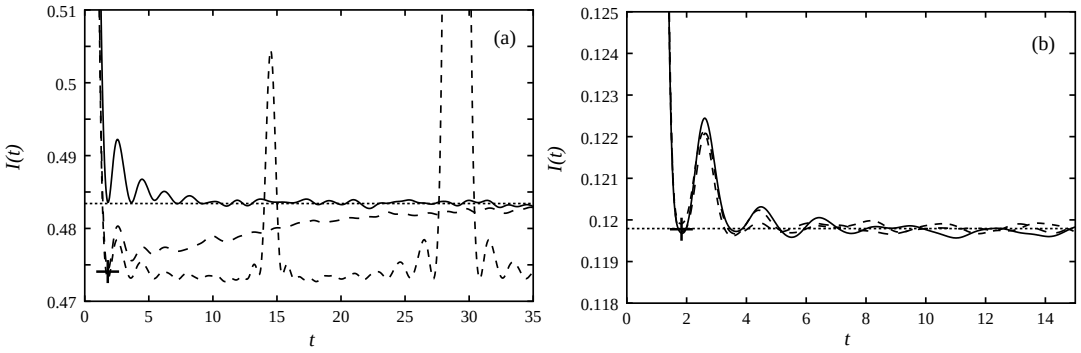

Fig. 1 – The purity $I(t)$ is shown for $m = n = 4$ in (a) and $m = 10, n = 50$ in (b), starting from an initially pure state as a function of time. The ensembles considered are: POE (solid line), GOE (long dashed line), picket fence (short dashed line). The value I_∞ (dotted horizontal line). The value at the first minimum $I(2\pi/\Gamma)$ (upright cross). We suppress the evolution at short times to see the difference between GOE and POE.

spectra such as the picket fence. In this case all correlations are known and we could proceed to calculate I(t), but as we never expect an exact picket fence, we shall give qualitative comments and include the corresponding results in the numerical calculations shown in the figures. We expect the first minimum to be at least as deep as for GOE statistics, but as in this case the formal limit is nearly reached it cannot be very different. On the other hand we do not obtain randomization so the purity, after a few oscillations, remains at this level until we reach Poincaré recurrence. For times of the order of the Heisenberg time we expect a very different behaviour; more precisely if $t = 2\pi/d$. In this case we have exact revival in the case of the picket fence, while we certainly expect nothing for the random spectra. In the case of GOE fluctuations what we have to consider is the width of the kth neighbour spacing distribution. It is known to increase logarithmically. For $k = 1$ we have a width of ≈ 1.25 and for $k = 8$ it is already ≈ 1.85. We thus will see no signature in this case. We should though note two facts: first we will find for a picket fence an additional partial revival at half the time mentioned, because of the terms with $2t$ in the time evolution; second and more importantly the long-range stiffness of spectra in low-dimensional systems [9] implies a saturation in the width of the kth neighbour spacing distribution and could therefore show recurrences. This point will not be addressed in the present letter.

We illustrate the time dependence discussed by the evolution of the purity for the cases $m = 4$, $n = 4$ and $m = 10$, $n = 50$, in the Figures 1(a) and (b) respectively. In both cases we show the results for spectra with random and GOE like fluctuations as well as picket fence spectra, using the same value of Γ in all cases. We notice that our expectations are well fulfilled. At the beginning all curves are equal, and the first minimum occurs at the same time. In Fig. 1a for $N = 16$ the minimum is lower for GOE like fluctuations and for Picket fence spectra than for random spectra, though the effect is quite small. This difference is negligible in Fig. 1b. with $N = 500$. For large times the GOE- and the Poisson-like fluctuations yield the same result even for small N. The results at all times coincide with our theoretical expectations. This is also true for the recurrences, which are only seen for picket fence spectra both at $t = 2\pi/d$ and more weakly at $t = \pi/d$.

In Fig. 1a the rise of purity after the first minimum follows roughly that of the Fourier transform of the two-point function with appropriate scaling. This is not surprising, because we may expect that a cluster expansion of the correlations relevant for the difference from the

random case is dominated by the two-point function. A detailed calculation will be reserved for a later publication.

We have proposed a random matrix approach to decoherence, which allows to isolate the effects of spectral statistics, *i.e.* the universal features of chaos and integrability on this process. We find the behaviour of the purity at short and at intermediate times is dominated by the width of the wave packet in energy space, which is the one property of the packet we have to take into account even in our models. It determines both the prefactor of the initial quadratic behaviour and the position of the first minimum. The plateau reached at long times is dominated by the number of levels of the smaller subsystem involved, but the corrections resulting from the number of levels of the larger system are also important. Both are again depending on the width of the packet, but in this case the ratio of these numbers is decisive. We find correction terms of order $1/N$ or larger, that are important for narrow wave packets. While the plateau does not depend on integrability or chaos, the intermediate time scales show a weak effect of chaos that is reminiscent of the correlation hole.

For the strong coupling case we conclude, that the influence of spectral statistics on decoherence is small and limited to packets with few eigenstates. This implies a very limited direct impact of chaos on decoherence. Yet for specific choices of the wave packets this situation may be very different, if these relate to the invariant tori of the integrable system in Hilbert space, as we know from the behaviour of intensities [8]. For chaotic systems strong deviations from this theory may still occur if the packet had some special relation to features known as scars, that may be associated with only slightly unstable periodic orbits, parabolic manifolds, or bifurcations that influence the chaotic dynamics at larger scales. Due to the mixed phase space used in [3] a comparison with the calculations presented there is delicate, but the general picture is not incompatible. Calculations in other entirely chaotic and integrable systems are under way.

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